ESSAY IV. APPLICATION OF THE PREDICTION PROCESS TO MARTINGALES

0. INTRODUCTION.

Let $X(t)$, $t \geq 0$, be a right-continuous supermartingale relative to an increasing family of σ-fields $\mathcal{G}_t^*$ on some probability space $(\Omega^*, \mathcal{F}^*, P^*)$. We assume that the $\mathcal{G}_t^*$ are countably generated for each t. It is then easy, by using indicator functions of generators of $\mathcal{G}_t^*$, to construct a sequence $X_{2(n+1)}(t)$, $1 \leq n$, of real-valued processes such that $\{X(s), (X_{2(n+1)}(s)), s \leq t\}$ generates $\mathcal{G}_t^*$ for each rational t. We can now transfer both process and probability to the canonical space Ω of Essay 1. We simply set $P\{w_{2n-1}(s) = 0$, all $s \geq 0$ and $n \geq 1\} = 1$, and for $S \in \times_{t \geq 0} \bar{B}_\infty$ (see Essay 1, Section 1 for notation)

$$P\{(w_{2n}(\cdot)) \in S\} = P^*\{(X(\cdot), X_{2(n+1)}(\cdot)) \in S\} .$$

Then we obtain a canonically defined process $X_t((w_n)) = w_2(t)$ which is a supermartingale with respect to P and the σ-fields $\mathcal{G}_t^o$ of Essay 1. In the present work, we let X_t denote this process (rather than the sequential process (w_{2n})), and we drop the odd coordinates from the notation (i.e., we discard the set of probability 0 where they are non-zero). Thus we do not allow any "hidden information": $\mathcal{F}_t^o = \mathcal{G}_t^o$. By a well-known convergence theorem we have

$$\begin{aligned} E(X_{s+t} | \mathcal{F}_{t+}^o) &= \lim_{r \to t+} E(X_{s+t} | \mathcal{F}_r^o) \\ &\leq \lim_{r \to t+} X_r \\ &= X_t . \end{aligned}$$

Hence X_t is a supermartingale relative to $\mathcal{F}_{t+}^o$, and we can connect it with its prediction process Z_t^P.

As in Essay 1, the method requires that P be treated as a variable. In the present work we are concerned initially with three familiar classes of P on $(\Omega, \mathcal{F}^o)$, as follows.