ESSAY III. CONSTRUCTION OF STATIONARY STRONG-MARKOV TRANSITION PROBABILITIES

Let X_t be a continuous parameter stochastic process on $(\Omega, \mathcal{F}, P)$ with values in a metrizable Lusin space $(E, \mathcal{E})$ (i.e., $\mathcal{E}$ is the Borel σ-field of a Borel set E in a compact metric space $\bar{E}$). In order just to state the property of X_t that it be a "time-homogeneous Markov process", it is necessary to introduce some form of conditional probability function to serve as transition function. From an axiomatic standpoint it is of course desirable to assume as little as possible about this function. An interesting and difficult problem is then to deduce from such assumptions the existence of a complete Markov transition probability $p(t,x,B)$ for (P, X_t) which satisfies the Chapman-Kolmogorov identities

$$(1.1) \qquad p(s+t,x,B) = \int p(s,x,dy)p(t,y,B) ,$$

thus giving rise to a family $(P^x, x \in E)$ of Markovian probabilities for which

$$(1.2) \qquad P^x(X_{s+t} \in B | \sigma(X_\tau, \tau \leq s)) = P^{X_s}\{X_t \in B\} .$$

The analogous time-inhomogeneous problem (of obtaining a $p(s,x;s+t,B)$) was treated by J. Karush (1961), and considerably later the present problem was taken up by J. Walsh [9]. It seems, however, that for the homogeneous case the solution remained complicated and conceptually difficult.[1]

Since the publication of these two works, a new tool has appeared on the scene which has an obvious bearing on the problem, namely, the "prediction process" of [5] and [8]. Accordingly, the present essay aims to show what can be done by using this method. But it is not simply a question of applying a new device. Our view is that the prediction process is fundamental to the problem, and the hypotheses which are needed to apply it give a basic understanding of the nature of the difficulties. A suggested way of viewing the entire matter is as follows. The prediction process is in some sense the best approximation to X_t by a process which does have a

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[1]The hypotheses of Theorem 3 of [9] are ultimately consequences of ours (Corollary 1.9 below).