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INVARIANT MEASURES ON STIEFEL MANIFOLDS WITH APPLICATIONS TO MULTIVARIATE ANALYSIS

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Let $V_{k,m}$ denote the Stiefel manifold which consists of $m \times k (m \geq k)$ matrices X such that $X'X = I_k$. We present decompositions of a random matrix X and then of the invariant measure on $V_{k,m}$, relative to a fixed subspace ν in R^m, for all possible four cases to be considered according to the sizes of $k, m,$ and the dimension of ν. The results are utilized for deriving the distributions of the canonical correlation coefficients between two random matrices of "general" dimensions, and for discussing high dimensional limit theorems (as $m \to \infty$) on $V_{k,m}$.

1. Introduction. We consider the Stiefel manifold $V_{k,m}$ which consists of $m \times k(m \geq k)$ matrices X such that $X'X = I_k$, the $k \times k$ identity matrix. For $k = m$, the Stiefel manifold is the orthogonal group $O(m)$. An invariant measure (i.m.) on $V_{k,m}$ is given by the differential form (d.f.)

$$(X'dX) = \bigwedge_{i<j}{}_1^k \boldsymbol{x}_j' d\boldsymbol{x}_i \bigwedge_{j=1}^{m-k} \bigwedge_{i=1}^{k} \boldsymbol{b}_j' d\boldsymbol{x}_i, \tag{1.1}$$

in terms of the exterior products ($\wedge$), where we choose an $m \times (m-k)$ matrix B such that $[X \vdots B] = (\boldsymbol{x}_1 \cdots \boldsymbol{x}_k \vdots \boldsymbol{b}_1 \cdots \boldsymbol{b}_{m-k}) \in O(m)$ and $d\boldsymbol{x}$ is an $m \times 1$ vector of differentials. The volume of $V_{k,m}$ is given by $w(k,m) = 2^k \pi^{km/2}/\Gamma_k(m/2)$, where $\Gamma_k(a) = \pi^{k(k-1)/4} \prod_{i=1}^k \Gamma(a - (i-1)/2)$, and the normalized i.m. of unit mass on $V_{k,m}$ is denoted by $[dX](= (X'dX)/w(k,m))$.

The Grassmann manifold $G_{k,m-k}$ consists of k-planes, i.e., k-dimensional linear subspaces in R^m. For $X \in V_{k,m}$, we can write $X = GQ$; that is, X in $V_{\kappa,m}$ is determined uniquely by the specification of the k-plane, i.e., the "reference" matrix G in $G_{\kappa,m-k}$ and the orientation $Q \in O(k)$ of G. An i.m.

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