Stochastic Inequalities
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ON FKG–TYPE AND PERMANENTAL INEQUALITIES

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In this paper we survey results from Rinott and Saks (1990) on a "2m-function" inequality which generalizes the FKG and associated inequalities. We also present related conjectures and partial results on permanents and sums of permutation matrices. We hope that the motivation given in the first part of the paper, and the subsequent discussion will attract the attention of problem solvers to our conjectures.

1. Introduction

The FKG inequality (Fortuin, Kasteleyn and Ginibre (1971)) has been applied in many fields, including statistical mechanics, combinatorics, reliability theory and stochastic inequalities. In order to state it we need the following notation and definition: for $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_k)$ in $\mathbb{R}^k$, $\mathbf{x} \vee \mathbf{y}$ and $\mathbf{x} \wedge \mathbf{y}$ in $\mathbb{R}^k$ are defined to have coordinates $(\mathbf{x} \vee \mathbf{y})_j = \max(x_j, y_j)$ and $(\mathbf{x} \wedge \mathbf{y})_j = \min(x_j, y_j)$, $j = 1, \ldots, k$.

DEFINITION A σ-finite (nonnegative) measure μ on $\mathbb{R}^k$ is said to be an *FKG measure* if μ has a density function ϕ with respect to some product measure $d\sigma$ on $\mathbb{R}^k$, (that is, $d\sigma(\mathbf{x}) = \Pi_{j=1}^k d\sigma_j(x_j)$, and $d\mu(\mathbf{x}) = \phi(\mathbf{x})d\sigma(\mathbf{x})$), satisfying for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$,

$$(1) \qquad \phi(\mathbf{x})\phi(\mathbf{y}) \leq \phi(\mathbf{x} \vee \mathbf{y})\phi(\mathbf{x} \wedge \mathbf{y}).$$

Condition (1) is referred to as *multivariate total positivity of order 2* (MTP_2) in Karlin and Rinott (1980). It can be shown that if a positive density ϕ is TP_2 in every pair of variables, then (1) holds, i.e., ϕ is MTP_2. We now state the FKG inequality as follows:

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