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LOWER BOUNDS ON MULTIVARIATE DISTRIBUTIONS WITH PREASSIGNED MARGINALS

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It is well known that the Fréchet lower bound on bivariate distributions with given marginals, F_1 and F_2, given by

$$\max\{F_1(x_1) + F_2(x_2) - 1, 0\},$$

cannot be extended for the case of three or more dimensions. To overcome this difficulty, in order to arrive at a sharp lower bound for multivariate distributions with preassigned marginals, we introduce the concept of the moment of inertia of a multivariate distribution about a given line in $\mathbb{R}^n$ and construct the distribution with the maximal moment of inertia about the line corresponding to the lower Fréchet bound. The multinormal case is discussed in some detail.

1. Introduction

In this paper we suggest an n-variate extension to the Fréchet lower bound for bivariate cumulative distribution functions (c.d.f.s). Recall that for $\Pi(F_1, F_2)$, the class of bivariate c.d.f.s with marginals F_1 and F_2, the Fréchet lower bound is defined as

$$H_*(x, y) = \max\{F_1(x) + F_2(y) - 1, 0\}$$

This does not lend itself to any straight-forward extension to the case of $\Pi(F_1, F_2, \ldots, F_n)$ when $n > 2$ where $\Pi(F_1, F_2, \ldots, F_n)$ is the class of all c.d.f.s whose univariate marginals are the c.d.f.s $F_1, F_2, \ldots, F_n$. However, by observing that the Fréchet upper bound for this class,

$$H^*(x_1, x_2, \ldots, x_n) = \min\{F_1(x_1), \ldots, F_n(x_n)\},$$

concentrates all the density on the curve

$$\{(x_1, \ldots, x_n) | F_1(x_1) = \cdots = F_n(x_n)\},$$

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