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COVARIANCE SPACES FOR MEASURES ON POLYHEDRAL SETS

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Let V be a given subset of $\mathbb{R}^n$. We are interested in determining the associated moment space $\mathcal{C}_r[V]$. The latter consists of all points $\mathbf{c} = (c(\mathbf{i}); |\mathbf{i}| \leq r)$ which can be realized as $c(\mathbf{i}) = \int x^{\mathbf{i}}\mu(dx)$, for all $|\mathbf{i}| \leq r$, by a measure μ on V. Here, $\mathbf{i} = (i_1, \ldots, i_n)$ with $i_j \in \mathbf{Z}_+$ and $|\mathbf{i}| = i_1 + \cdots + i_n$. Let $\mathcal{C}_r(V)$ be the analogous homogeneous moment space $\mathcal{C}_r(V)$, where one insists on $|\mathbf{i}| = r$. The calculation of $\mathcal{C}_r[V]$ is shown to be equivalent to that of $\mathcal{C}_r(W)$, with W as a suitable affine imbedding of V into $\mathbb{R}^{n+1}$. A central role is played by the dual $\mathcal{C}_r(V)^*$ of the convex cone $\mathcal{C}_r(V)$. One may interpret $\mathcal{C}_r(V)^*$ as the set of all homogeneous polynomials $f(x) = f(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ of degree r that are nonnegative on V.

Detailed results are given only for the important case $r = 2$. Let $\mathcal{Q}_n$ be the linear space of all symmetric $n \times n$ matrices, supplied with the natural inner product $(A, B) = \mathrm{Tr}(AB)$. The pair $\mathcal{C}_2(V)$ and $\mathcal{C}_2(V)^*$ has a natural interpretation as a pair of dual convex cones in $\mathcal{Q}_n$. In fact, $\mathcal{C}_2(V)^*$ is the set of all $Q \in \mathcal{Q}_n$ such that $x^tQx \geq 0$ for all $x \in V$. Special attention is given to the second order moment spaces $\mathcal{C}_2(K)$ and $\mathcal{C}_2[T]$ with

$$K = \{x \in \mathbb{R}^n : Ax \geq 0\}; \qquad T = \{x \in \mathbb{R}^n : Bx + e \geq 0\}.$$

Here A and B denote given $m \times n$ matrices. Our description of the latter moment spaces involves the crucial cone $\mathcal{P}_m = \{Q \in \mathcal{Q}_m : x^tQx \geq 0 \text{ for all } x \in \mathbb{R}_+^m\}$.

These results are quite explicit when $m \leq 4$, as happens, for instance, when T is a triangle in $\mathbb{R}^2$ or a simplex in $\mathbb{R}^3$. This is largely due to the very simple structure of the cone $\mathcal{P}_m$ in the case $m \leq 4$, due to Diananda (1962). The remaining problem, of determining the second order moment spaces $\mathcal{C}_2(K)$ or $\mathcal{C}_2[T]$ for the case $m \geq 5$, is essentially equivalent to the long standing difficult open problem to determine the precise structure of the cone $\mathcal{P}_m$ when $m \geq 5$. Concrete applications will be given in subsequent papers.

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