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MATRIX EXTREMES AND RELATED STOCHASTIC BOUNDS

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The spaces $F_{n\times k}$ and S_k^+ consisting of rectangular and positive definite matrices are developed as partially ordered sets having lower and upper bounds. Under Loewner (1934) ordering, spectral lower and upper bounds are constructed for pairs $\mathbf{A},\mathbf{B} \in (S_k^+, \succeq_L)$ and are shown to be tight. Similar bounds are given for pairs in $(F_{n\times k}, \succeq)$ in terms of singular decompositions under an induced ordering. Applications pertaining to $(S_k^+, \succeq_L)$ include stochastic bounds for distributions of quadratic forms, minimal dispersion bounds in certain regular ensembles, and bounds on the peakedness of certain weighted vector sums. Applications to $(F_{n\times k}, \succeq)$ support the uniform improvement of any pair of first-order experimental designs.

1. Introduction

Extremal problems persist throughout applied probability and statistics. Their solutions often shed new light on structural aspects of the system at hand.

To fix ideas, we reexamine the concentration properties of measures $\mu(\cdot;\mathbf{p})$ induced by weighted sums $\sum_{i=1}^n p_i X_i$ of *iid* random scalars $\{X_1,\ldots,X_n\}$ having a symmetric log-concave density. Here $\mathbf{p} = [p_1,\ldots,p_n]$ satisfies $\{0 \le p_i \le 1, p_1 + \cdots + p_n = 1\}$, and we let $F(t;\mathbf{p}) = \mu([-t,t];\mathbf{p})$ with $t > 0$. Proschan (1965) has shown for each $t > 0$ that $F(t;\mathbf{p})$ is order-reversing under majorization, *i.e.*, if $\mathbf{p}$ majorizes $\mathbf{q}$, then $F(t;\mathbf{q}) \ge F(t;\mathbf{p})$ and thus $\mu(\cdot;\mathbf{q})$ is more peaked than $\mu(\cdot;\mathbf{p})$ in the sense of Birnbaum (1948).

Since linear functions arise in a variety of contexts not entailing ordered weights, we pose the further question: If neither $\mathbf{p}$ majorizes $\mathbf{q}$ nor $\mathbf{q}$ majorizes $\mathbf{p}$, what then may be said regarding the concentration properties of $\mu(\cdot;\mathbf{p})$ and $\mu(\cdot;\mathbf{q})$? One answer follows immediately on observing that the ordered simplex supporting majorization is a lattice with greatest lower

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