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PARTITIONING INEQUALITIES IN PROBABILITY AND STATISTICS[1]

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This article surveys fair–division or cake–cutting inequalities in probability and statistics, including bisection inequalities, basic fairness inequalities, convexity tools, superfairness inequalities, and partitioning inequalities in hypotheses testing and optimal stopping theory. The emphasis is measure theoretic, as opposed to game theoretic or economic, and a number of open problems in the area are mentioned.

1. Introduction

The main purpose of this article is to present a unified study of a class of partitioning inequalities in the theories of probability and statistics; it is not meant to be a complete review of the subject. The emphasis is measure theoretic with emphasis on both constructive (algorithmic) and non–constructive techniques, including generalizations of classical "cake–cutting" inequalities, the ham sandwich theorem, and classical statistical problems such as Fisher's Problem of the Nile, the problem of smiliar regions, and the classification problem.

The overall framework is as follows. There are a finite number of (countably additive) probability measures $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ defined on the *same* measurable space $(\Omega, \mathcal{F})$, and a class Π of $\mathcal{F}$–measurable partitions of Ω is specified. (Recall that $(A_i)_1^k$ is an (ordered) $\mathcal{F}$–partition of Ω if $\cup_1^k A_i = \Omega, A_i \cap A_j = \emptyset$ for $i \neq j$, and $A_i \in \mathcal{F}$ for all i.) From this collection of partitions Π a single partition is sought which will satisfy some objective such as bisection or minimax–risk property. It may help the reader to keep in mind either a cake–cutting or a hypotheses–testing interpretation of this setting throughout the following sections.

In the cake–cutting interpretation, Ω is a cake which must be divided among n people having values $\mu_1, \ldots, \mu_n$ (that is, $\mu_j(A)$ is the relative value

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