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ORDERINGS ARISING FROM EXPECTED EXTREMES, WITH AN APPLICATION

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We bound the expected maximum order statistics $\{EX_{(n)}\}_{n=1}^{\infty}$ of a d.f. F_X both above and below. Our results have an interpretation in terms of stochastic orderings $\leq_e$ and $\leq_{we}$ defined as follows: $F_X \leq_e F_Y$ iff $EX_{(n)} \leq EY_{(n)}$ for all n, and $F_X \leq_{we} F_Y$ iff $EX_{(n)} \leq EY_{(n)}$ for n sufficiently large. We apply our results on $\leq_{we}$ to the end-to-end delay in a resequencing $M/G/\infty$ queue.

1. Introduction

If $X_1, \ldots, X_n$ are i.i.d. random variables with parent distribution F_X, let $X_{(n)}$ denote the maximum order statistic $\max(X_1, \ldots, X_n)$. We are interested in the case when F_X has nonnegative lower endpoint, and upper endpoint $+\infty$. In this case we wish to control the behavior of $X_{(n)}$ as $n \to \infty$; in particular, to bound it above and below in expectation or in related senses. The bounds should be as free of assumptions on the distribution F_X as possible.

Our original motivation for investigating this question was the study of stochastic models arising in computing (Downey and Maier (1990)). There the X_i are interpreted as time delays. (See Section 3 for a typical example, a resequencing $M/G/\infty$ queueing model.) But our results have a more general interpretation, in terms of stochastic inequalities. If a relation $\leq_e$ and its weak counterpart $\leq_{we}$ are defined on the class of finite-mean distributions of nonnegative r.v.'s by

$$(1) \qquad F_X \leq_e F_Y \iff EX_{(n)} \leq EY_{(n)}, \quad n \geq 1$$

$$(2) \qquad F_X \leq_{we} F_Y \iff EX_{(n)} \leq EY_{(n)}, \quad n \text{ suff. large}$$

then our results have implications for $\leq_e$ and $\leq_{we}$.

The orderings $\leq_e$ and $\leq_{we}$ are very natural, but seem never to have been studied before. Chan (1967) showed that a distribution is uniquely

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