Stochastic Orders and Decision under Risk
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MULTIVARIATE STOCHASTIC ORDERINGS AND GENERATING CONES OF FUNCTIONS

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Let C be a convex cone of functions $\phi : \mathbb{R}^n \to \mathbb{R}$; let X and Y be random vectors. Write $X \leq_C^{st} Y$ to mean $E\phi(X) \leq E\phi(Y)$ for all functions ϕ in C such that the expectations exist. Familiar examples of stochastic orderings that take this form are obtained from the convex cones of (1) increasing functions, (2) convex functions, (3) Schur-convex functions, and (4) centrally symmetric quasiconcave functions. In the literature, various properties of the corresponding orderings have been given, mostly on a case by case basis. The purpose of this paper is to gain some understanding of how some of these properties arise directly as consequences of conditions satisfied by the underlying convex cone C. A collection of examples is given.

1. Introduction. For a class C of (measurable) functions $\phi : \mathbb{R}^n \to \mathbb{R}$, and for random vectors X and Y, this paper is concerned with the condition

$$E\phi(X) \leq E\phi(Y) \text{ for all functions } \phi \in C \text{ such that the expectations exist.} \tag{1.1}$$

Write $X \leq_C^{st} Y$ to mean that (1.1) holds, or equivalently, when X has distribution F and Y has distribution G, write $F \leq_C^{st} G$.

Orderings of the form $\leq_C^{st}$ constitute a large class of what are sometimes called *stochastic orderings*. Other possible approaches to stochastic orderings are reviewed by Mosler and Scarsini (1991), but in this paper, only the definition via (1.1) is considered.

If (1.1) holds, then it is immediate that C can be replaced by the smallest convex cone containing C, so in this paper it is usually assumed from the start that C is a convex cone.

A number of orderings of the form $\leq_C^{st}$ have been defined and studied in the literature, usually with C specified, but sometimes with considerable generality. Some of these orderings are based upon the cone of functions isotone

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