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A PATHWISE APPROACH TO DYNKIN GAMES*

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Abstract

We reduce the classical discrete-time game of optimal stopping between two players, known as "Dynkin game", to a *pathwise* (deterministic) game of timing, by addition of a suitable non-adapted compensator (λ_n) to the payoff. This compensator satisfies $\mathbb{E}(\lambda_n|\mathcal{F}_n) \equiv 0$, where $\mathcal{F}_n$ is the information available to the players at time $t = n$, and $\mathbb{E}$ denotes expectation with respect to the underlying probability measure $\mathbb{P}$; the compensator also enforces the *non-anticipativity constraint* that the strategies of both players be stopping times of $(\mathcal{F}_n)$. A pair of such stopping times is identified, which leads to a saddle-point for each of these games; and it is shown that the value V of the stochastic game is obtained by "averaging out" the value $W(\omega)$ of the pathwise game: $V = \int_\Omega W(\omega)\mathbb{P}(d\omega)$.

1. Introduction and Summary. We present a simple approach to the discrete-time *stochastic game of optimal stopping* (or timing) known as "Dynkin game" (Dynkin & Yushkevich (1968)), with payoff from player **A** to player **B** equal to

$$\mathcal{R}(\sigma,\tau) = U_\sigma 1_{\{\sigma<\tau\}} + L_\tau 1_{\{\tau<T,\tau\leq\sigma\}} + \xi 1_{\{\sigma=\tau=T\}}. \tag{1.1}$$

Here $U_n \geq L_n (n\epsilon\mathbb{N}_0)$ are integrable random sequences, adapted to the filtration $\mathbb{F} = \{\mathcal{F}_n, n\epsilon\mathbb{N}_0\}; \sigma$ and τ are stopping times of $\mathbb{F}$ with values in $\{0,1,\ldots,T\}$, at the disposal of players **A** and **B**, respectively; $T \leq \infty$ is the "horizon" of the game; ξ is an integrable random variable; and

$$\overline{V} \triangleq \inf_\sigma \sup_\tau \mathbb{E}\mathcal{R}(\sigma,\tau), \qquad \underline{V} \triangleq \sup_\tau \inf_\sigma \mathbb{E}\mathcal{R}(\sigma,\tau) \tag{1.2}$$

are the upper- and lower- values, respectively, of the game. Notice that in the infinite-horizon case ($T = \infty$) we are allowing stopping times to be extended-valued, i.e., to take the value $+\infty$.

Under reasonably mild conditions, we show that this game has value $V = \overline{V} = \underline{V}$, as well as a saddle-point of stopping times $(\hat{\sigma},\hat{\tau})$, by looking

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