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PARAMETRIZING DOUBLY STOCHASTIC MEASURES

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Doubly stochastic measures can be identified with the trace of a pair of (Lebesgue) measure preserving maps of the unit interval to itself.

It has been of traditional interest in probability theory to produce a random vector (or metric space element), which has a given distribution and is defined on a standard space, such as $[0, 1]$ endowed with Lebesgue measure. In a classic work, Lévy (1937, section 23) used an approach based on conditioning. For the purpose of the Skorokhod representation, Billingsley (1971, Theorem 3.2) considered the case of random elements of a general metric space. Whitt (1976, Lemma 2.7) considered general measures on R^n and employed a Borel isomorphism to treat questions of extremal correlation and minimal variance. Rüschendorf (1983) used a similar approach to consider a general class of optimization problems.

In this note, we revisit the question of representing a random element in the special case of a doubly stochastic measure on the unit square. First we show the existence of a random element (using essentially Whitt's approach) with a refinement to a *canonical* representation. Then we turn to criteria for extremality of a doubly stochastic measure. Our aim is to provide the reader who is interested in extremality with different settings.

This paper was invited for presentation at the AMS-IMS-SIAM Joint Summer Research Conference on Distributions with Fixed Marginals, Doubly-Stochastic Measures, and Markov Operators, July 31–August 6, 1993. At the Conference, the author learned of *A Representation for Doubly Stochastic Measures* by W. F. Darsow and E. T. Olsen. Subsequently, a revision (Darsow and Olsen, 1994) was provided to the author. Interested readers are directed to this work for variant arguments and related topics, including copulas.

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