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The Marginal Problem in Arbitrary Product Spaces

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Using a property of perfect measures due to Marczewski and Ryll-Nardzewski (1953), we unify the solutions to the marginal problem for two-dimensional products. We then extend that property to arbitrary product spaces and provide a general solution to the marginal problem in arbitrary product spaces. Our results remove the restrictive topological assumptions in earlier works and are valid in spaces where the σ-algebras need not be countably generated. A general result on the existence of simultaneous preimage measures as well as a "converse" to it are derived.

1. Introduction. Let $\{(X_i, \mathcal{A}_i, P_i), i \in I\}$ be a family of probability spaces. The marginal problem is connected to probabilities P on the space $(\prod_{i\in I} X_i, \otimes_{i\in I} \mathcal{A}_i)$ such that P has the given family $\{P_i, i \in I\}$ as marginals, i.e. $P \circ \pi_i^{-1} = P_i$ for every $i \in I$ where $\pi_j : \prod_{i\in I} X_i \to X_j$ is the canonical projection map. It can be formulated as:

> Given $S \subset \prod_{i\in I} X_i$, what are the conditions that would ensure the existence of a probability P on $(\prod_{i\in I} X_i, \otimes_{i\in I}\mathcal{A}_i)$ with marginals $\{P_i, i \in I\}$ such that $P^*(S) = 1$?

For $I = \{1, 2\}$, with underlying spaces being Polish, this problem reduces to Strassen's (1965) marginal problem. Variants of this have been investigated earlier by Banach (1948), Marczewski (1948, 1951) for the case when projections are required to be stochastically independent under P and later for the general case by Kellerer (1964a,b) as well as Maharam (1971). The importance of the above formulation in applications can be found in Hoffman-Jørgensen (1987).

In recent work by Hansel and Troallic (1978, 1986), Shortt (1983) and Kellerer (1984, 1988) solutions to Strassen's version were derived under topological restrictions. Plebanek (1989) has combined the solution for the finitely

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