Distributions with Fixed Marginals and Related Topics
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Copulas, Marginals, and Joint Distributions

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Any pair of univariate marginal distributions can be combined with any copula to yield a bivariate distribution with the given marginals. This being the case, it is tempting to conclude that the dependence properties of the distribution can be determined by examination of the copula alone. Unfortunately, this idea is seriously flawed: (i) Copulas exist which yield the Fréchet upper bound for some marginal pairs and the Fréchet lower bound for other marginal pairs. (ii) There is no nonconstant measure of dependence which depends only on the copula. (iii) Weakly convergent sequences of bivariate distributions with continuous marginals exist for which the unique corresponding copulas do not converge. Related issues are considered.

1. Introduction. If C is a bivariate distribution function with marginals uniform on $[0,1]$, and if F and G are univariate distribution functions, then as is well known,

$$H = C(F, G) \tag{1.1}$$

is a bivariate distribution function with marginals F and G. In this context, C is variously called a "dependence function" or a "copula", and C is often thought of as a function which "couples" the marginals F and G. A number of parametric families of copulas have been proposed in the literature, with at least an implied suggestion that they be used to generate bivariate distributions with given marginals through the formula (1.1). What can one say about the joint distributions so generated if the marginals are unknown?

The marginals F and G can be inserted into any copula, so they carry no direct information about the coupling; at the same time, any pair of marginals can be inserted into C so C carries no direct information about the marginals. This being the case, it may seem reasonable to expect that the connections

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