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IN QUEST OF BIRKHOFF'S THEOREM IN HIGHER DIMENSIONS

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A doubly stochastic matrix is a non-negative function defined on $\{1,\ldots,m\}\times\{1,\ldots,m\}$ such that all row and column sums are 1. A hypermatrix is a non-negative function defined on a set of the form $X=\{1,\ldots,m_1\}\times\ldots\times\{1,\ldots,m_n\}$. A hypermatrix is called multiply stochastic if it satisfies a suitably generalized version of the row and column condition for ordinary doubly stochastic matrices. Note that this is a double generalization of doubly stochastic matrices: we not only consider higher dimensions but allow "non-square matrices". Our goal is to describe extremal multiply stochastic hypermatrices in terms of their support, in terms of transfer vectors, and as local minima of the entropy function and to characterize the set of such extremals for a certain class of $3\times3\times3$ hypermatrices.

1. Introduction. An $n\times n$ matrix (m_{ij}) is called doubly stochastic if $m_{ij}\geq 0$ for all $i,j=1,\ldots,n$ and

$$\sum_{i=1}^{n} m_{ij}=1 \quad \text{for} \quad j=1,\ldots,n$$

and

$$\sum_{j=1}^{n} m_{ij}=1 \quad \text{for} \quad i=1,\ldots,n.$$

A doubly stochastic matrix is called extremal if it is an extremal element in the set of all doubly stochastic matrices. In other words, a doubly stochastic matrix is extremal if it cannot be represented as a convex combination of other doubly stochastic matrices. Garrett Birkhoff (1946) proved the following characterization of extremal doubly stochastic matrices. It is known that the theorem was later rediscovered independently by John von Neumann, but he has not published his proof.

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