Distributions with Fixed Marginals and Related Topics
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PROXIMITY OF PROBABILITY MEASURES WITH COMMON MARGINALS IN A FINITE NUMBER OF DIRECTIONS

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We provide estimates of the closeness between probability measures defined on $\mathbb{R}^n$ which have the same marginals in a finite number of arbitrary directions. Our estimates show that the probability laws get closer in the λ-metric which metrizes the weak topology when the number of coinciding marginals increases. Our results offer a solution to the computer tomography paradox stated in Gutmann, Kemperman, Reeds, and Shepp (1991).

1. Introduction and Statement of the Problem. Let Q_1 and Q_2 be a pair of probabilities, i.e. probability measures defined on the Borel σ-field of $\mathbb{R}$. Lorentz (1949) gave criteria for the existence of a probability density function $g(\cdot)$ on $\mathbb{R}^2$ taking only two values, 0 or 1, and having Q_1 and Q_2 as marginals. Kellerer (1961) generalized this result, obtaining the necessary and sufficient conditions for the existence of a density $f(\cdot)$ on $\mathbb{R}^2$ which satisfies the inequalities $0 \leq f(\cdot) \leq 1$, and has Q_1 and Q_2 as marginals (see also Strassen (1965) and Jacobs (1978)). Fishburn et al. (1990) were able to show that Kellerer's and Lorentz's conditions are equivalent, i.e. for any density $f(\cdot), 0 \leq f \leq 1$, on $\mathbb{R}^2$ there exists a density $g(\cdot)$ taking the values 0 and 1 only, which has the same marginals. In general, similar results hold for probability densities on $\mathbb{R}^m, \quad m \geq 2$, when the $(m-1)$-dimensional marginals are prescribed. A considerably stronger result was established by Gutmann et al. (1991). This is that, for any probability density $f(\cdot), 0 \leq f \leq 1$, on $\mathbb{R}^m$ and for any finite number of directions, there exists a probability density $g(\cdot)$ taking the values 0, 1 only, which has the same marginals as $f(\cdot)$ in the chosen directions. It follows that densities having the same marginals in a finite number of arbitrary directions may differ considerably in the uniform

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