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Ambiguity in Bounded Moment Problems

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Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $\mathcal{K}$ be a linear subspace of $\mathcal{L}_1(\mu)$ (e.g. generated by the one-dimensional projections, if X is a product space). The following inverse problem is treated: To what extent is a set $A \in \mathcal{A}$ "$\mathcal{K}$-determined" within the class of all (fuzzy sets) $g \in \mathcal{L}_\infty(\mu)$ satisfying $0 \leq g \leq 1$, i.e. which lower and upper bounds A_* and A^* for A can be derived from knowing the integrals $\int_A f d\mu$, $f \in \mathcal{K}$ – thus generalizing the uniqueness problem ($A_* = A^*$).

Introduction. This is the natural extension of a paper entitled "Uniqueness in bounded moment problems," Kellerer (1993). The questions treated there had their origin in a central problem of tomography: which n-dimensional objects can be reconstructed from the measures of all their $(n-1)$-dimensional sections orthogonal to the different axes? In this form the planar case was studied by Kuba and Volcic (1988), while the extension to higher dimensions was carried out by Fishburn et al. (1990, 1991).

As was done by Kemperman (1990, 1991), the author in Kellerer (1993) subsumed this classical case under the following "bounded moment problem": given a measure space $(X, \mathcal{A}, \mu)$ and a family $\mathcal{K}$ of integrable test functions, which sets $A \in \mathcal{A}$ are – up to null sets – uniquely determined by the integrals $\int_A f d\mu$, $f \in \mathcal{K}$? In the weak version A is compared with ordinary sets only, while in the strong version this comparison takes place in the class $\mathcal{G}$ of all fuzzy sets (i.e. functions attaining their values in the unit interval). Since both models coincide in important situations, but methods of convex analysis cannot be applied to the weak version, in the sequel the strong version will be emphasized.

Now the search for uniquely determined sets is in fact a very restricted view of this kind of inverse problems. As considered by Kuba and Volcic (1993) in the planar case, also in the general case two sets of uniqueness are associated

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