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Doubly Stochastic Measures: Three Vignettes

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The first section contains a nonstandard analysis proof of Strassen's theorem. The second section contains a uniqueness theorem for doubly stochastic measures on a family of supports that give rise in a natural way to certain dynamical systems. In the final section, we consider the action of $SL_2(\mathbb{Z})$ on $C(G \times G)^*$, G a compact abelian group, and study those orbits that consist entirely of stochastic or doubly stochastic measures.

0. Introduction. Our present state of knowledge concerning extreme doubly stochastic measures on $I \times I$ ($I = [0,1]$) and doubly stochastic measures with prescribed support is meager compared with the the situation for doubly stochastic finite matrices.

Recall that according to the Birkhoff-von Neumann Theorem, Birkhoff (1946), the extreme doubly stochastic $n \times n$-matrices are precisely the $n \times n$-permutation matrices. This being so, a set $S \subseteq \mathbf{n} \times \mathbf{n}$ ($\mathbf{n} = \{1, \ldots, n\}$) contains the support of a doubly stochastic matrix iff S contains the support of a permutation matrix iff the bipartite graph whose incidence relation S describes admits a perfect matching. P. Hall's theorem, Lovasz and Plummer (1986), tells us that S fails to admit a perfect matching iff S is disjoint from a set of the form $A \times B$, $A, B \subseteq \mathbf{n}$, $|A| + |B| > n$. With the availability of highly efficient algorithms for bipartite matching, Lovasz and Plummer (1986), our understanding of doubly stochastic finite matrices may be considered quite satisfactory.

One measure-theoretic analogue for the permutation matrices might be the class of doubly stochastic measures supported on graphs of measure preserving transformations. Alas, these form merely a proper subclass of the extreme doubly stochastic measures, so the most straightforward generalization of the Birkhoff-von Neumann Theorem is false. Indeed, extreme doubly

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