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NONSQUARE "DOUBLY STOCHASTIC" MATRICES

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An $n \times m$ non-negative matrix with uniform row sum m and column sum n is called a "doubly stochastic" matrix. When $n = m$, such a matrix is a scale multiple of a doubly stochastic matrix in its classical sense. Garrett Birkhoff proved a theorem characterizing all classical extremal doubly stochastic matrices as permutation matrices. We will discuss the characterization of the extremal matrices for nonsquare "doubly stochastic" matrices in the spirit of Birkhoff's theorem.

An $n \times m$ matrix $\mathbf{M} = (m_{ij})$ is called a *doubly stochastic* matrix (with uniform marginals) of size $n \times m$ if

$$\text{(uniform row marginals)} \qquad \sum_{j=1}^{m} m_{ij} = m \quad \text{for} \quad i = 1, 2, ..., n. \tag{1}$$

$$\text{(uniform column marginals)} \qquad \sum_{i=1}^{n} m_{ij} = n \quad \text{for} \quad j = 1, 2, ..., m. \tag{2}$$

$$\text{(positivity)} \qquad m_{ij} \geq 0 \quad \text{for} \quad i = 1, 2, ..., n \quad \text{and} \quad j = 1, 2, ..., m. \tag{3}$$

For example,

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_2 = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{4}$$

are two doubly stochastic matrices of size 3×4.

For integers $m, n \geq 1$, let $\mathcal{M}_{n \times m}$ denote the set of all doubly stochastic matrices of size $n \times m$. Then it is easy to see that $\mathcal{M}_{n \times m}$ is a convex set (of

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