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Random Variables, Distribution Functions, and Copulas – A Personal Look Backward and Forward

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The author recalls his initial involvement with the basic notions of probability theory, which began in the late forties in the context of number theory, continued through his work with B. Schweizer on probabilistic metric spaces, and culminated in a correspondence with Fréchet that led to the identification and naming of copulas. The author speculates about possible future applications of the theory of distribution functions with given margins: In particular, there is the prospect of productive treatment of situations where, say, no common probability space can be found for a given set of "random variables," but such common probability spaces exist for arbitrary proper subsets of the given set.

This paper presents my recollections of, and outlook on, one phase of the development of our subject, up to the early sixties. It also indicates what I believe to be a promising direction for future work. I thank the organizers of the conference for the opportunity to do this, and I thank the referees for their very helpful comments.

My serious engagement with probability began in the late forties, in the context of number theory, where one meets statements such as: "almost all numbers are composite" [Hardy-Wright (1960), p. 8] and: "The probability that a number should be quadratfrei is $6/\pi^2$" [ibid, p. 267; "quadratfrei" is now usually anglicized to "squarefree"]. They refer to a function, often denoted by δ and called the "density," defined for certain sets of positive integers by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \#(A \cap \{1, 2, \cdots, n\})$$

whenever the limit exists, where $\#(S)$ denotes the number of elements in the (finite) set S. But δ is not a measure, since its domain is not closed under the binary Boolean operations: as pointed out in Niven (1951), p. 424 "sequences A and B can be constructed [Buck (1946), p. 571] so that $\delta(A)$ and $\delta(B)$ exist but $\delta(A \cup B)$ and $\delta(A \cap B)$ do not."

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