Efficient estimation in a semiparametric heteroscedastic autoregressive model

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This paper characterizes and constructs efficient estimators of the autoregression parameter in the heteroscedastic autoregression model of order 1 with unknown innovation density and unknown volatility function.

1. Introduction. In this paper I consider a stationary and ergodic semiparametric heteroscedastic autoregressive model of order 1. More precisely, I assume that the observations $X_0, X_1, X_2, \ldots, X_n$ of this model satisfy the structural relation

$$X_t = \rho X_{t-1} + \sigma(X_{t-1})\varepsilon_t, \quad t = 1, 2, \ldots, n$$

for some real parameter ρ, some Lipschitz-continuous positive function σ that is bounded away from zero, and innovations $\varepsilon_1, \ldots, \varepsilon_n$ which are independent of the initial observation X_0 and are independent and identically distributed (iid) with common positive density γ that has zero mean, variance 1, finite fourth moment and finite Fisher information for location and scale. The latter means that γ is absolutely continuous and

$$\int (1 + x^2) \left(\frac{\gamma'(x)}{\gamma(x)} \right)^2 \gamma(x)\, dx < \infty.$$

I also assume that

$$\rho^2 + \limsup_{|x| \to \infty} \frac{\sigma^2(x)}{1 + x^2} < 1. \tag{1.1}$$

This condition yields the (geometric) ergodicity of the model as shown by Maercker [10]. Her results establish V-uniform ergodicity with $V(x) = 1 + x^2$, $x \in \mathbb{R}$.

In what follows ρ, σ and γ are assumed to be unknown. The goal is to estimate ρ efficiently in the presence of the infinite-dimensional nuisance parameter (σ, γ). One possible estimator of ρ is the least squares estimator

$$\hat{\rho}_n^{LS} = \frac{\sum_{j=1}^n X_{j-1} X_j}{\sum_{j=1}^n X_{j-1}^2}.$$

This estimator is $n^{1/2}$-consistent as $n^{1/2}(\hat{\rho}_n^{LS} - \rho)$ has a limiting normal distribution with mean zero and variance $E[X_1^2 \sigma^2(X_1)]/(E[X_1^2])^2$. If σ were known one could use the weighted least squares estimator

$$\hat{\rho}_n^{WLS} = \frac{\sum_{j=1}^n X_{j-1} X_j / \sigma^2(X_{j-1})}{\sum_{j=1}^n X_{j-1}^2 / \sigma^2(X_{j-1})}.$$

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