A Simple Low-Bias Estimate Following a Sequential Test With Linear Boundaries

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We suppose a Brownian motion with drift and linear stopping boundaries, possibly truncated, is observed. The maximum likelihood estimate of the drift, upon reaching a boundary, is known to be badly biased, while the uniformly minimum variance unbiased estimate of Liu and Hall (1998, *Sequential Analysis* **17**: 91-107) is difficult to compute. A bias-adjusted estimate of Whitehead (1986, *Biometrika* **73**: 573-581) is also complex computationally and may still have substantial bias. We propose a modification of the maximum likelihood estimate, a *segmented estimate*, which has a simple explicit formula. Computation shows the estimate to have little bias and to have a competitive mean-square-error. The results apply to various sequential testing problems through asymptotic approximation and adjustment for discrete-time observation.

1. Introduction. We consider observation of a Brownian motion $X(t)$, observed in continuous time t, with a stopping boundary. The boundary is typically chosen to provide a test of hypotheses about the drift parameter. But the test is unimportant here; we focus on estimating the drift once a boundary has been reached, and without regard to the conclusion of the sequential test.

We assume a *linear stopping boundary*, consisting of an upper boundary (a line with a positive intercept), a lower boundary (a line with a negative intercept), and possibly a vertical boundary. The boundary forms a closed region with one possible exception: parallel upper and lower boundaries without a vertical boundary are allowed. These boundaries include Wald's *sequential probability ratio tests* (SPRTs) [20], Anderson's *triangular designs* [1] (see also [17, 10, 22]), and Armitage's [2] *restricted designs*; these are the only fully-sequential designs (continuously distributed stopping times) for which the distribution of the stopped Brownian motion along the boundary is explicitly available. Formulas were derived by Anderson [1]; we use the simpler versions of [11].

This Brownian motion paradigm serves as an asymptotic approximation to many sequential analysis problems, including random sampling from a parametric model and a proportional-hazards survival analysis staggered-entry two-arm clinical trial model; see [22, 13]. Whitehead [22] includes a boundary adjustment to allow for observation in discrete-time. Recently, sequential tests with triangular boundaries have been utilized in clinical trials, e.g., [18], [19], and [3], in AIDS, cardiology and pediatrics, respectively.

Whitehead [21] considered estimation of the drift upon hitting the stopping boundary, at (t, x), say, with $x = X(t)$. The maximum likelihood estimate (MLE) is x/t. He found it to have considerable bias (noted earlier by Cox [5]; see his Figure 1, showing the bias function for a symmetric SPRT and a symmetric triangular test (2-SPRT).