# Asymptotic Design of General Triangular Stopping Boundaries for Brownian Motion *

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We consider triangular stopping boundaries for a Brownian motion with drift, with specified error probabilities at two given values for the drift. We consider the Kiefer-Weiss problem of finding boundaries which minimize the maximum expected stopping time asymptotically as the error probabilities tend to zero. A construction is given which minimizes the objective function through fourth order optimality. This extends earlier work for the simpler symmetric (equal error probabilities) case, where fifth order minimization was achieved.

**1. Introduction.** Consider testing the hypotheses $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ for the drift $\theta$ of a Brownian motion $Y$. Kiefer and Weiss [12] suggest searching for the test such that the maximum (over $\theta$) of the average stopping time (AST) is minimized under some prespecified error probabilities $(\alpha, \beta)$ at $\theta_0$ and $\theta_1$. Lorden [14] combined two SPRTs, of $\theta_0$ versus $\theta_m$ and of $\theta_0$ versus $\theta_m$ for some intermediate $\theta_m$, to form a particular class of tests called 2-SPRTs. He showed, for any fixed $\theta^*$, a 2-SPRT can be chosen such that its stopping time $T^*$ satisfies

$$E_{\theta^*}T^* = \inf_{T \in D(\alpha,\beta)} E_{\theta^*}T + o(1)$$

as $\min(\alpha, \beta) \to 0$, where $D(\alpha, \beta)$ is the class of all tests with error probability bounds $(\alpha, \beta)$. For Brownian motion, 2-SPRTs have triangular stopping boundaries.

In the symmetric case when $\beta = \alpha$, it is known that $\sup_\theta E_\theta T = E_{\theta_m} T$ for all $T \in D(\alpha, \alpha)$, where $\theta_m = (\theta_0 + \theta_1)/2$. Hence the 2-SPRT stopping time $T_m$ with respect to this $\theta_m$ satisfies

$$\sup_\theta E_\theta T_m = \inf_{T \in D(\alpha,\alpha)} \sup_\theta E_\theta T + o(1).$$

Lai [13] also showed that, in the symmetric case, the asymptotic shape of the minimax (Kiefer-Weiss) stopping boundaries are triangular. In the asymmetric case, Huffman [11] extended Lorden's results to show that by solving $\tilde{\theta}$ from some equation numerically, the stopping time $\tilde{T}$ of 2-SPRT with respect to this $\tilde{\theta}$ satisfies

$$\sup_\theta E_\theta \tilde{T} = \inf_{T \in D(\alpha,\beta)} \sup_\theta E_\theta T + o(|\log \alpha|^{1/2})$$

as $\alpha \to 0, \beta \to 0$ and $0 < C_1 < \log \alpha / \log \beta < C_2 < +\infty$, where $C_1$ and $C_2$ are constants. Note that $|\log \alpha|^{1/2} \to \infty$ as $\alpha \to 0$. Such results were extended further by Dragalin and Novikov [3]. They showed that

$$\sup_\theta E_\theta \tilde{T} = \inf_{T \in D(\alpha,\beta)} \sup_\theta E_\theta T + O(1)$$

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