DERIVATIVE IN THE MEAN OF A DENSITY AND STATISTICAL APPLICATIONS

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In the parametric model $\mathcal{P} = \{P_\theta : \theta \in \boldsymbol{\Theta}\}$ with p-dimensional parameter θ, such that P_θ are mutually absolutely continuous with densities differentiable in the mean, we prove an identity transforming the derivative in the mean of the likelihood ratio in $\mathcal{P}$ into the derivative in the mean of a general statistic. This identity makes possible, among others, to approximate the expectation (or other moments) of a statistic, locally in a neighborhood of θ_0 and non-asymptotically under a finite number of observations. This in turn provides the local power of a general test of the simple hypothesis $\mathbf{H}_0$: $\theta = \theta_0$. Using these results, we show that the classical χ^2-test is the locally most powerful invariant test for the hypothesis of balanced multinomial trials.

1. Introduction

Consider the parametric model $(\mathcal{X}, \mathcal{B}, \mathcal{P})$, where $\mathcal{P} = \{P_\theta : \theta \in \boldsymbol{\Theta}\}$ is a family of probability distributions on $\mathcal{B}$ with $\boldsymbol{\Theta}$ being an open subset of $\mathbb{R}^p$. Assume that the members of the family $\mathcal{P}$ are mutually absolutely continuous with square roots of densities differentiable in quadratic mean. More precisely, we impose the following conditions on $\mathcal{P}$:

A.1 The probabilities P_θ, $\theta \in \boldsymbol{\Theta}$ are mutually absolutely continuous. Denote $f_\theta(\cdot) = dP_\theta/dP_{\theta_0}$ the density of P_θ with respect to P_{θ_0}, $\theta \in \boldsymbol{\Theta}$, with a fixed $\theta_0 \in \boldsymbol{\Theta}$.

A.2 The mapping $\theta^* \mapsto f_{\theta^*}/f_\theta$ is P_θ-differentiable in the mean for all $\theta, \theta^* \in \boldsymbol{\Theta}$ with derivative $\mathbf{l}(\cdot,\theta) = \big(\ell_1(\cdot,\theta), \ldots, \ell_p(\cdot,\theta)\big)'$ such that $\mathbb{E}_\theta\|\mathbf{l}(\cdot,\theta)\|^2 < \infty$ if there exists a random vector $\mathbf{l}(\cdot,\theta)$ satisfying

$$(1.1) \qquad \mathbb{E}_\theta\left\{\sum_{k=1}^{p}\left|\frac{1}{h_k}\left(\frac{f_{\theta+\mathbf{h}}}{f_\theta} - 1\right) - \ell_k(\cdot,\theta)\right|\right\} \to 0 \quad \text{as } \mathbf{h} \to \mathbf{0}.$$

Then, obviously

$$(1.2) \qquad \mathbb{E}_\theta\big(\mathbf{l}(\cdot,\theta)\big) = \lim_{\mathbf{h}\to 0} \mathbb{E}_\theta\left\{\frac{1}{|\mathbf{h}|}\left(\frac{f_{\theta+h}}{f_\theta} - 1\right)\right\} \equiv 0.$$

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