Fractional Brownian motion as a differentiable generalized Gaussian process

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Abstract

Brownian motion can be characterized as a generalized random process and, as such, has a generalized derivative whose covariance functional is the delta function. In a similar fashion, fractional Brownian motion can be interpreted as a generalized random process and shown to possess a generalized derivative. The resulting process is a generalized Gaussian process with mean functional zero and covariance functional that can be interpreted as a fractional integral or fractional derivative of the delta-function.

Keywords: Brownian motion, fractional Brownian motion, fractional derivative, covariance functional, delta function, generalized derivative, generalized Gaussian process

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1 Introduction

Fractional Brownian motion, like ordinary Brownian motion, has almost everywhere continuous sample paths of unbounded variation and ordinary derivatives of the process do not exist. Gel'fand and Vilenkin (1964) provided an alternative characterization of Brownian motion as a generalized Gaussian process defined as a random functional on a space of well behaved functions. Interpreted as a generalized random process, Brownian motion is differentiable.

A generalized Gaussian process is uniquely determined by its mean functional and the bivariate covariance functional. Correspondingly, the generalized derivative of a Gaussian process with zero mean functional is a generalized Gaussian process with zero mean functional and covariance functional that can be computed from the covariance functional of the original process. Gel'fand and Vilenkin provide a description of the generalized Gaussian process which represents the derivative of Brownian motion. This process has a covariance functional that can be interpreted in terms of the delta-function.

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