# Singular Stochastic Control in Optimal Investment and Hedging in the Presence of Transaction Costs

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**Abstract**

In an idealized model without transaction costs, an investor would optimally maintain a proportion of wealth in stock or hold a number of shares of stock to hedge a contingent claim by trading *continuously*. Such continuous strategies are no longer admissible once proportional transaction costs are introduced. The investor must then determine when the stock position is sufficiently "out of line" to make trading worthwhile. Thus, the problems of optimal investment and hedging become, in the presence of transaction costs, singular stochastic control problems, characterized by instantaneous trading at the boundaries of a "no transactions" region whenever the stock position falls on these boundaries. In this paper, we review various formulations of the optimal investment and hedging problems and their solutions, with particular emphasis on the derivation and analysis of Hamilton-Jacobi-Bellman (HJB) equations using the dynamic programming principle. A particular numerical scheme, based on weak convergence of probability measures, is provided for the computation of optimal strategies in the problems we consider.

# 1 Introduction

The problems of optimal investment and consumption and of option pricing and hedging were initially studied in an idealized setting whereby an investor incurs no transaction costs from trading in a market consisting of a risk-free asset ("bond") with constant rate of return and a risky asset ("stock") whose price is a geometric Brownian motion with constant rate of return and volatility. For example, Merton (1969, 1971) showed that, for an investor acting as a price-taker and seeking to maximize expected utility of consumption, the optimal strategy is to invest a constant proportion (the "Merton proportion") of wealth in the stock and to consume at a rate proportional to wealth. In the related problem of option pricing and hedging, arbitrage considerations of Black and Scholes (1973) demonstrated that, by setting up a portfolio of stock and option that is risk-free, the value of an option must equal the amount of initial capital required for this hedging.

However, both the Merton strategy and the Black-Scholes hedging portfolio require continuous trading and result in an infinite turnover of stock in any finite