Brownian Motion and the Classical Groups

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Abstract

Let Γ be chosen from the orthogonal group O_n according to Haar measure, and let A be an $n \times n$ real matrix with non-random entries satisfying $TrAA^t = n$. We show that $TrA\Gamma$ converges in distribution to a standard normal random variable as $n \to \infty$ uniformly in A. This extends a theorem of E. Borel. The result is applied to show that if entries $\beta_1, \cdots, \beta_{k_n}$ are selected from Γ where $k_n \to \infty$ as $n \to \infty$, then $\sqrt{\frac{n}{k_n}} \sum_{j=1}^{[k_n t]} \beta_j, 0 \leq t \leq 1$ converges to Brownian motion. Partial results in this direction are obtained for the unitary and symplectic groups.

Keywords: Brownian motion; sign-symmetry; classical groups; random matrix; Haar measure

1 Introduction

Let O_n be the group of $n \times n$ orthogonal matrices, and let Γ be chosen from the uniform distribution (Haar measure) on O_n. There are various senses in which the elements of $\sqrt{n}\Gamma$ behave like independent standard Gaussian random variables to good approximation when n is large.

To begin with, a classical theorem of Borel [6] shows that $P\{\sqrt{n}\Gamma_{11} \leq x\} \to \Phi(x)$ where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$. Theorems 2.1 and 2.2 below refine this, showing that an arbitrary linear combination of the elements of Γ is approximately normal: as $n \to \infty$,

$$\sup_{\substack{A \neq 0 \\ -\infty < x < \infty}} \left| P\left\{\frac{Tr(A\Gamma)}{\sqrt{||A||/\sqrt{n}}} \leq x\right\} - \Phi(x) \right| \to 0. \tag{1.1}$$

Here A ranges over all non-zero $n \times n$ matrices and $||A|| = Tr(AA^t)$; thus the normal approximation result is uniform in A. Borel's theorem follows by taking A to have a one in the one-one position and zeros elsewhere. When A above is the identity matrix, Diaconis and Mallows (see [11]) proved that $Tr\Gamma$ is approximately normal; this follows by taking A as the identity. As A varies, it follows that linear combinations of elements of Γ are also approximately normal. Interpolating between these facts and Borel's result, we prove that linking appropriately normalized entries from Γ yields in the limit standard Brownian motion. This is stated precisely in Theorem 3 below.

We give a little history. Borel's result is usually stated thus: Let X be the first entry of a point randomly chosen from the n-dimensional unit sphere. Then $P\{\sqrt{n}X \leq x\} \to \Phi(x)$ as n tends to ∞. Since the first row(or column) of a uniformly chosen orthogonal matrix is uniformly distributed on the unit sphere, Theorem 2.1 includes Borel's theorem. Borel, following earlier work by Mehler [31] and Maxwell [28, 29], proved the result as a rigorous version of the equivalence of ensembles in statistical mechanics. This says that features of the