Variational formulas and explicit bounds of Poincaré-type inequalities for one-dimensional processes

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Abstract

This paper serves as a quick and elementary overview of the recent progress on a large class of Poincaré-type inequalities in dimension one. The explicit criteria for the inequalities, the variational formulas and explicit bounds of the corresponding constants in the inequalities are presented. As typical applications, the Nash inequalities and logarithmic Sobolev inequalities are examined.

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1 Introduction

The one-dimensional processes in this paper mean either one-dimensional diffusions or birth-death Markov processes. Let us begin with diffusions.

Let $L = a(x)\mathrm{d}^2/\mathrm{d}x^2 + b(x)\mathrm{d}/\mathrm{d}x$ be an elliptic operator on an interval $(0, D)$ $(D \le \infty)$ with Dirichlet boundary at 0 and Neumann boundary at D when $D < \infty$, where a and b are Borel measurable functions and a is positive everywhere. Set $C(x) = \int_0^x b/a$, here and in what follows, the Lebesgue measure $\mathrm{d}x$ is often omitted. Throughout the paper, assume that

$$Z := \int_0^D e^C/a < \infty. \tag{1.0}$$

Hence, $\mathrm{d}\mu := a^{-1}e^C\mathrm{d}x$ is a finite measure, which is crucial in the paper. We are interested in the first Poincaré inequality

$$\|f\|^2 := \int_0^D f^2\mathrm{d}\mu \le A\int_0^D f'^2 e^C := AD(f), \quad f \in \mathbb{C}_d[0, D],\ f(0) = 0, \tag{1.1}$$

where $\mathbb{C}_d$ is the set of all continuous functions, differentiable almost everywhere and having compact supports. When $D = \infty$, one should replace $[0, D]$ by $[0, D)$ but we will not mention again in what follows. Next, we are also interested in the second Poincaré inequality

$$\|f - \pi(f)\|^2 := \int_0^D (f - \pi(f))^2\mathrm{d}\mu \le \overline{A}D(f) \qquad f \in \mathbb{C}_d[0, D], \tag{1.2}$$

where $\pi(f) = \mu(f)/Z = \int f\mathrm{d}\mu/Z$. To save the notations, we use the same A (resp., $\overline{A}$) to denote the optimal constant in (1.1) (resp., (1.2)).

The aim of the study on these inequalities is looking for a criterion under which (1.1) (resp., (1.2)) holds, i.e., the optimal constant $A < \infty$ (resp., $\overline{A} < \infty$),

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