On Itô's Complex Measure Condition[1]

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Abstract

The complex measure condition was introduced by Itô (1965) as a sufficient condition on the potential term in a one-dimensional Schrödinger equation and/or corresponding linear diffusion equation to obtain a Feynman-Kac path integral formula. In this paper we provide an alternative probabilistic derivation of this condition and extend it to include any other lower order terms, i.e. drift and forcing terms, that may be present. In particular, under a complex measure condition on the lower order terms of the diffusion equation, we derive a representation of mild solutions of the Fourier transform as a functional of a jump Markov process in wave-number space.

Keywords: Duality, multiplicative cascade, multi-type branching random walk

1 Introduction

The *complex measure condition* was introduced by Itô (1965) as a sufficient condition on the potential term $\theta(x)$ in the one-dimensional Schrödinger equation

$$\frac{h}{i}\frac{\partial \phi}{\partial t} = \frac{h^2}{2m}\frac{\partial^2 \phi}{\partial x^2} - m\,\theta(x)\phi \tag{1.1}$$

for the so-called Feynman principle of quantization. More specifically, under the condition that $\theta(x)$ is the Fourier transform of a complex measure of bounded variation on $(-\infty, \infty)$, Itô (1965) establishes the validity of the Feynman-Kac path integral formula appropriate to (1.1). Throughout "complex measure" will imply a *regular measure* with *finite total variation* without further mention. Itô (1965) further notes that his method is also applicable to the linear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{a^2}{2}\frac{\partial^2 u}{\partial x^2} + c(x)u. \tag{1.2}$$

Our first encounter with the complex measure condition arose in efforts to better understand the branching random walk associated with incompressible Navier-Stokes equations that was originally developed by LeJan and Sznitman (1997) and elaborated upon in Bhattacharya et al (2002). From this point of view the binary branching tree structure associated with the nonlinear Navier-Stokes equation is replaced by a unary tree structure for the linear diffusion equation. However in preparing the present article we learned about variants of these results for the Schrödinger equation recently given by Kolokoltsov (2000, 2002) and in references therein. We have not found a specific reference to the extension to lower order terms given here, but given the rather sizeable physics

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