θ-expansions and the generalized Gauss map

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Abstract

Motivated by problems in random continued fraction expansions, we study θ-expansions of numbers in $[0, \theta)$ where $0 < \theta < 1$. For such a number θ, we study the generalized Gauss transformation defined on $[0, \theta)$ as follows:

$$T(x) = \begin{cases} \frac{1}{x} - \theta[\frac{1}{\theta x}] & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

One of the problems that concerns us is the symbolic dynamics of this map and existence of absolutely continuous invariant probability.

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1 Introduction

Suppose that μ is a probability on the real line. Consider the following law of motion: If you are at x pick a number Z according to the law μ and move to $Z + x$. Continue the motion with independent choices at each stage. This is nothing but the familiar random walk. Suppose that by an error the law is transcribed as : move to $Z + \frac{1}{x}$, then what happens? To make sense of the problem, from now on we consider the state space to be $(0, \infty)$. Let μ be a probability on $[0, \infty)$ which drives the motion. If you are at x move to $Z + \frac{1}{x}$ where Z is chosen independent of the past and has law μ. This leads us to the Markov process

$$X_0 = x > 0; \qquad X_{n+1} = Z_{n+1} + \frac{1}{X_n} \quad \text{for } n \geq 0$$

where $(Z_n; n \geq 1)$ is an i.i.d sequence of random variables, each having law μ. The purpose of the paper is to discuss this process.

2 Generalities

If μ is δ_0, the point mass at zero, then $X_n = x$ or $1/x$ according as n is even or odd. Unless $x = 1$ the process does not converge in distribution. For each $x > 0$, $\frac{1}{2}(\delta_x + \delta_{1/x})$ is an invariant distribution for the process. In fact any invariant probability is a mixture of these. If $\mu = \delta_a$ where $a > 0$, then the process starting at x is deterministic and is the sequence – in the usual notation of continued fractions – $[x;]$, $[a; x]$, $[a; a, x]$, $\cdots$ which converges to the number given by the continued fraction $[a; a, a, \cdots]$. We leave the easy calculation involving convergents to the interested reader. The point mass at this point is the unique