Ordered Triple Designs and Wreath Products of Groups

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Abstract

We explore an interesting connection between a family of incidence structures and wreath products of finite groups.

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1 Introduction

The problem discussed in this paper arose from a study in [2] of the set of primitive maximal subgroups of a finite symmetric group $\mathsf{Sym}\,\Omega$ containing a given subgroup of $\mathsf{Sym}\,\Omega$. Application of group theoretic results, depending on the classification of finite simple groups, reduced the problem of describing one family of such maximal subgroups to a problem concerning a certain kind of incidence structures. We chose this topic because of the unexpected links between several types of mathematical objects.

For a finite set Ω the maximal subgroups of $\mathsf{Sym}\,\Omega$ may be divided into several disjoint families: intransitive maximal subgroups, imprimitive maximal subgroups, and several families of primitive maximal subgroups; see [6]. A given permutation group G on Ω may be contained in many maximal subgroups of $\mathsf{Sym}\,\Omega$. The intransitive and imprimitive maximal overgroups of G may be determined from the G-orbits and the G-invariant partitions of Ω. However, determining the primitive overgroups of G is a difficult problem in general. It has been essentially solved in [6] and [9] in the case where G itself is primitive, and even this case required significant use of the finite simple group classification. In [2] we were concerned with a more general situation: the groups G of interest were innately transitive, in other words, they contain a minimal normal subgroup that is transitive. The maximal overgroups of G studied in [2] were wreath products in product action (see Section 3 for the definition of wreath products and product actions). Investigating such overgroups led to a study of certain incidence structures discussed in Section 2. Their connection with overgroups of innately transitive groups is described in more detail in Section 3, and a construction is given in Section 4.