

Diffusions on the Simplex from Brownian Motions on Hypersurfaces

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Abstract

The $(n - 1)$ -dimensional simplex is the collection of probability measures on a set with n points. Many applied situations result in simplex-valued data or in stochastic processes that have the simplex as their state space. In this paper we study a large class of simplex-valued diffusion processes that are constructed by first “coordinatising” the simplex with the points of a smooth hypersurface in such a way that several points on the hypersurface may correspond to a given point on the simplex, and then mapping forward the canonical Brownian motion on the hypersurface. For example, a particular instance of the Fleming-Viot process on n points arises from Brownian motion on the $(n - 1)$ -dimensional sphere. The Brownian motion on the hypersurface has the normalised Riemannian volume as its equilibrium distribution. It is straightforward to compute the corresponding distribution on the simplex, and this provides a large class of interesting probability measures on the simplex.

Keywords: manifold; stochastic differential equation; measure-valued process; compositional data; Riemannian volume element; Fleming-Viot process

1 Introduction

Many data sets come in the form of proportions that add to unity (that is, as points in a simplex with dimension one less than the number of proportions). For example, there is the breakdown of the composition of an ore sample into component minerals or the division of a family’s expenditures into housing, food, clothing, leisure, *etc.* This type of data is often referred to as *compositional* and a standard reference for models and inference in this area is [1].

Such data can also have a temporal component. For example, there are the proportions of the population at any time having each of the possible combinations of alleles of a given set of genes (see, for example, [5]). There appears to be something of a dearth of flexible, tractable models for such stochastic processes.

Of course, stochastic processes on the simplex are an elementary instance of processes taking values in the set of probability measures on an arbitrary measurable space. However, the literature in this more general area is primarily concerned with models