Chapter 8

Lecture 28

Example 7. X_i are iid uniformly over $(0, \theta)$ for $\theta \in \Theta = (0, \infty)$.

Homework 6

1. Show that

   a. With respect to Lebesgue measure on $\mathbb{R}^n$,

   $$\ell(\theta \mid s_n) = \begin{cases} 1/\theta^n & \text{if } \theta \geq X_i \ \forall i \\ 0 & \text{otherwise} \end{cases}$$

   and $\hat{\theta} = \max\{X_1, \ldots, X_n\}$.

   b. Condition 2 in the Theorem above is satisfied, and hence $\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta$ for all θ (which we check directly also); but the likelihood function is not continuous, and hence the information function is not defined.

   c. $E_\theta(\hat{\theta}_n) = \frac{n}{n+1}\theta$, and $\theta_n^* := \frac{n+1}{n}\hat{\theta}$ is unbiased.

   d. $n(\theta - \hat{\theta}_n)$ has the asymptotic distribution with density $\frac{1}{\theta}e^{-\frac{z}{\theta}}$ on $(0, \infty)$, and so $\hat{\theta}_n$ has a non-normal limiting distribution and $\hat{\theta}_n - \theta = O(1/n)$.

(In regular cases, $\hat{\theta}$ has a normal limiting distribution and $\hat{\theta}_n - \theta = O(1/\sqrt{n})$.)

Asymptotic distribution of $\hat{\theta}$ (θ real) in regular cases

$X = \{x\}$ (arbitrary), $\mathcal{C}$ is a σ-field on X, P_θ is a probability on $\mathcal{C}$ and $\theta \in \Theta$ for Θ an open interval in $\mathbb{R}^1$. $dP_\theta(x) = \ell(\theta \mid x)d\nu(x)$, with ν a fixed measure. Let $s_n = (X_1, \ldots, X_n) \in S^{(n)} = X \times \cdots \times X$, $\mathcal{A}^{(n)} = \mathcal{C} \times \cdots \times \mathcal{C}$ and $P_\theta^{(n)} = P_\theta \times \cdots \times P_\theta$ on $\mathcal{A}^{(n)}$. We assume that $\ell(\theta \mid x) > 0$, $L(\theta \mid x) = \log_e \ell(\theta \mid x)$ has at least two continuous derivatives, $E_\theta\big(L'(\theta \mid x)\big) = 0$ and

$$I_1(\theta) = E_\theta\big(L'(\theta \mid x)\big)^2 = -E_\theta\big(L''(\theta \mid x)\big) > 0.$$