Chapter 5

Lecture 16

Example 1(e). We have X_i iid $ae^{-b(x-\theta)^4}$, with $a, b > 0$ chosen so that this is a density and $\mathrm{Var}_\theta(X_i) = 1$. Then

$$\ell_\theta(s) = \varphi_0(s)\exp\Big\{b\Big[\Big(4\sum_{i=1}^n X_i^3\Big)\theta - 6\Big(\sum_{i=1}^n X_i^2\Big)\theta^2 + 4\Big(\sum_{i=1}^n X_i\Big)\theta^3\Big] + A(\theta)\Big\},$$

which is not a one-parameter exponential family. It is called a "curved exponential family".

Sufficient conditions for the Cramér-Rao and Bhattacharya inequalities

As usual, we have $(S, \mathcal{A}, P_\theta)$, $\theta \in \Theta$, where Θ is an open subset of $\mathbb{R}^1$. μ is a fixed measure on S and $dP_\theta(s) = \ell_\theta(s)d\mu(s)$.

Condition 1. $\ell_\theta(s) > 0$ and $\delta \mapsto \ell_\delta(s)$ has, for each $s \in S$, a continuous derivative $\delta \mapsto \ell'_\delta(s)$. Let

$$\gamma_\theta^{(1)}(s) = \frac{\ell'_\theta(s)}{\ell_\theta(s)} = L'_\theta(s).$$

Condition 2. Given any $\theta \in \Theta$, we may find an $\varepsilon = \varepsilon(\theta) > 0$ such that $E_\theta(m_\theta^2) < +\infty$, where

$$m_\theta(s) = \sup_{|\delta-\theta|\le\varepsilon} |\gamma_\delta^{(1)}(s)|$$

– i.e., $m_\theta \in V_\theta$, which implies that $I(\theta) = E_\theta(\gamma_\theta^{(1)})^2 < +\infty$.

Condition 3. $I(\theta) > 0$.

12E Exact statement of Cramér-Rao inequality: Under conditions 1–3 above, if U_g is non-empty, then g is differentiable and

$$\mathrm{Var}_\theta(t) \ge \frac{(g'(\theta))^2}{I(\theta)} \;\forall\theta \in \Theta, t \in U_g.$$