Chapter 2

Lecture 6

Bayes estimation

We have the setup from the previous lecture: $(S, \mathcal{A}, P_\theta)$, $\theta \in \Theta$. We want to estimate $g(\theta)$. Let $\mathcal{B}$ be a σ-field in Θ and λ a probability on $\mathcal{B}$. We assume that g is $\mathcal{B}$-measurable and that $g \in L^2(\Theta, \mathcal{B}, \lambda)$ – i.e., that $\int_\Theta g(\theta)^2 d\lambda(\theta) < \infty$. We regard θ as a random element and $P_\theta(A)$ as a conditional probability that $s \in A$, given θ. Let $w = (s, \theta)$, $\Omega = S \times \Theta$ and $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ be the smallest σ-field containing all sets of the form $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We assume that $\theta \mapsto P_\theta(A)$ is $\mathcal{B}$-measurable for all $A \in \mathcal{A}$.

Lemma 1. *There exists a unique probability measure M on $\mathcal{C}$ such that*

$$M(A \times B) = \int_B P_\theta(A) d\lambda(\theta) \ \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

(λ is the distribution of θ, P_θ is the distribution of s given θ and M is the joint distribution of $w = (s, \theta)$. We will see the explicit formula for Q_s – the distribution of θ given s – soon.)

Consider an estimate t. Our assumption on g is that

$$E(g^2) = \int_\Omega g^2 dM = \int_\Theta g^2 d\lambda < \infty,$$

so

$$\begin{aligned} \overline{R}_t = \int_\Theta R_t(\theta) d\lambda(\theta) &= \int_\Theta E_\theta\big(t - g(\theta)\big)^2 d\lambda(\theta) \\ &= E\big(E((t-g)^2 \mid \theta)\big) = E(t-g)^2 = ||t-g||^2 \end{aligned}$$

(the norm taken in $L^2(\Omega, \mathcal{C}, M)$). We would like to choose a t to minimize this quantity. Since t is a function only of s, the desired minimizing estimate – which we will denote by t^* – is the projection of g to the subspace of all $\mathcal{A}$-measurable functions $t(s)$ satisfying $E_M\big(t(s)^2\big) < \infty$. We know that $t^*(s) = E\big(g(\theta) \mid s\big)$.