SOME CONVERSE LIMIT THEOREMS
FOR EXCHANGEABLE BOOTSTRAPS

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The bootstrap Glivenko-Cantelli and bootstrap Donsker theorems of Giné and Zinn (1990) contain both necessary and sufficient conditions for the asymptotic validity of Efron's nonparametric bootstrap. In the more general case of exchangeably weighted bootstraps, Praestgaard and Wellner (1993) and Van der Vaart and Wellner (1996) give analogues of the sufficiency half of the Theorems of Giné and Zinn (1990), but did not address the corresponding necessity parts of the theorems. Here we establish a new lower bound for exchangeably weighted processes and show that the necessity half of the a.s. bootstrap Glivenko-Cantelli theorem holds for exchangeably weighted bootstraps. We also make some progress toward the conjectured necessity parts of the bootstrap Donsker theorems with exchangeable weights.

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1 Introduction

Giné and Zinn (1990) established several beautiful limit theorems for Efron's nonparametric bootstrap of the general empirical process. One of the key tools used by Giné and Zinn (1990) was the *multiplier inequality* used earlier in Giné and Zinn (1983) with Gaussian multipliers to "Gaussianize" the empirical process and to relate the Gaussianized process to the symmetrized empirical process, and hence to the empirical process itself via symmetrization and de-symmetrization inequalities.

Efron's nonparametric bootstrap, which involves resampling from the empirical measure $\mathbb{P}_n$, can be viewed as one instance of an exchangeably weighted bootstrap with the weights being the components of a random vector which has a Multinomial distribution with n cells, n trials, and vector of "success" probabilities $(1/n, \ldots, 1/n)$. The first limit theory for exchangeably weighted bootstraps was established by Mason and Newton (1992); they treated exchangeable bootstrapping of the mean and of the classical empirical process. Praestgaard and Wellner (1993) extended the direct half

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