ON ORDER STATISTICS CLOSE TO THE MAXIMUM

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We investigate the asymptotic properties of order statistics in the immediate vicinity of the maximum of a sample. The usual domain of attraction condition for the maximum needs to be replaced by a continuity condition. We illustrate the potential of the approach by a number of examples.

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1 Introduction

Let $X_1, X_2, \ldots, X_n$ be a sample from X with distribution F. For convenience, we assume that $F(x) = P(X \leq x)$ is ultimately continuous for large values of x. We denote the order statistics of the sample by

$$X_1^* \leq X_2^* \leq \ldots \leq X_n^*.$$

Under the *(extremal domain of) attraction condition*, we mean the condition on F inducing the convergence in distribution of the normalized maximum $(X_n^* - b_n)/a_n$ to a non-degenerate limit law. Here a_n are positive constants while the constants b_n, $(n = 1, 2, \ldots)$ are real. The attraction condition can be given in terms of the *tail quantile function* $U(y) := inf\{x : F(x) \geq 1 - \frac{1}{y}\}$ as shown by de Haan [10]. F belongs to an extremal domain of attraction if and only if there exists an ultimately positive *auxiliary function* g and a real *extremal index* γ such that for all $y > 0$ the condition

$$\lim_{x \to \infty} \frac{U(xy) - U(x)}{g(x)} = \int_1^y w^{\gamma - 1}\, dw =: h_\gamma(y) \tag{1}$$

holds. In particular, $h_0(y) = \log y$. The constants can then be taken as $b_n = U(n)$ and $a_n = g(n)$ while the one-parameter family of possible limit laws is given by the *(class of) extreme value distributions*

$$G_\gamma(x) := \exp -(1 + \gamma x)_+^{-\frac{1}{\gamma}}.$$