A NONPARAMETRIC ASYMPTOTIC VERSION OF THE CRAMÉR-RAO BOUND

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The paper presents a genuinely asymptotic version of the Cramér-Rao bound, replacing the assumption of unbiasedness by locally uniform asymptotic unbiasedness, and the bound for the variance by a bound for the asymptotic variance. Bounds of this type are useful to obtain asymptotic results for estimator sequences which do not necessarily converge to a limit distribution. Under a condition slightly stronger than LAN, the minimal asymptotic variance obtained from the Convolution Theorem for regular estimator sequences turns out to be also a bound for the asymptotic variance of estimator sequences which are asymptotically unbiased, uniformly on shrinking χ^2-neighbourhoods. For nonparametric models with a convergence rate slower than $n^{1/2}$, the asymptotic variance of such estimator sequences is necessarily infinite.

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1 Introduction

The Cramér-Rao bound is one of the standard topics in textbooks on mathematical statistics, ranging from elementary to advanced. In view of its limited applicability this is hard to explain. The question whether a given unbiased estimator has minimal variance can be answered by the Cramér-Rao bound only in one particular case: If the family is exponential, say $p(x, \vartheta) = c(\vartheta) \exp[\vartheta T(x)]$, and if the functional to be estimated is $\vartheta \to \int T(x) P_\vartheta(dx)$ (see Müller-Funk et al. (1989) for minimal regularity conditions). In all other cases the Cramér-Rao bound is not attainable, hence not a suitable standard for judging the optimality of an unbiased estimator.

Some authors make a point of the fact that the Cramér-Rao bound can be attained *asymptotically.* However, conditions under which the Cramér-Rao bound is a bound for the asymptotic variance are of a totally different nature, and so are the proofs.

There is no straight way from the Cramér-Rao bound to a bound for the asymptotic variance. This can be seen from examples showing the following properties. (i) For every sample size there exists an unbiased estimator with minimal convex risk, (ii) the sequence of these estimators is asymptotically normal with a variance larger than the Cramér-Rao bound, (iii) there exists