A STATISTICAL APPROACH TO THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION

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We study the problem of estimating an unknown solution of the Cauchy problem for the Laplace equation, with L_2-norm loss, when the initial conditions are observed in a white Gaussian noise with a small spectral density. It is shown in particular that asymptotically minimax estimators are as a rule nonlinear.

AMS subject classifications: 62G05, 62G20.
Keywords and phrases: ill-posed problems, statistical approach, asymptotically minimax estimator.

1 Introduction

Hadamard (1912) proposed the famous example of the ill-posed boundary value problem

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0, \quad u(x,0) = 0, \quad u'_y(x,0) = \varphi(x). \tag{1}$$

He noticed that $\sup_x |\varphi_n(x)| \to 0$, as $n \to \infty$ for $\varphi_n(x) = n^{-1}\sin(2\pi nx)$ whereas *sup*-norm of the solution $u_n(x,y) = \sinh(2\pi ny)\sin(2\pi nx)/(2\pi n^2)$ tends to infinity for any $y > 0$. So the problem is called ill-posed in the Hadamard sense. Nevertheless this problem is the important geophysical problem of interpreting the gravitational or magnetic anomalies (see Lavrentiev (1967) and Tarchanov (1995)).

The usual approach to ill-posed problems deals with the recovery of a solution based on a "noisy" data. In order to guarantee consistent recovering some additional information about the function $\varphi(x)$ is required. It is assumed as a rule that φ belongs to a compact set $\mathcal{K}$ in a suitable space. The performance of the optimal solution depends on $\mathcal{K}$ and on the definition of the noisy data. Usually (see Tichonov & Arsenin (1977), Engl & Groetsch (1987)) it is assumed that the observed data are

$$\varphi^\varepsilon(x) = \varphi(x) + n^\varepsilon(x),$$

[1]The research was supported by NSF grant DMS-9600245 and DMS-9971608.