Estimating Relative Density on a Metric Space

James MacQueen
University of California, Los Angeles

Abstract. Let $X_1, X_2, \ldots,$ be stationary and ergodic random variables with values in a metric space M with distance d, let $P(A) = P(X_n \in A)$ and let $S(x,r) = \{y \in M : d(x,y) \leq r\}$. Let M_0 be the set of x for which $P(S(x,r)) > 0$ if $r > 0$, and suppose also that for x in M_0, $P(S(x,r))$ is continuous in x and is differentiable in r for $r \geq 0$, and with a positive derivative for all r in a neighborhood of 0. Consider the set M^* of pairs (x,y) such that both x and y are in M_0 and $\lim_{r \to 0} P(S(x,r))/P(S(y,r))$ exists and is a finite positive number $R(x,y)$. Then $R(x,y)$ is called the *relative density* of P for the pair x, y.

The differentiability condition is essentially the same as required for P to have a positive density in the Euclidean case. Note there may be pairs of elements (x,y) such that that $\lim_{r \to 0} P(S(x,r))/P(S(y,r)$ fails to exist, is zero, or is $+\propto$. For example, if P on the square $[0,1]X[0,1]$ concentrates a total probability of .5 uniformly on the line $x = y, 0 \leq x, y \leq 1$, and distributes probability uniformly on the square excepting this line, then the line of pairs $x = y$ is in M^* and so is the square excepting the line. But a pair with one element on the line and the other off gives a limit of 0 or $+\propto$ depending of which element appears in the numerator (or denominator). These kinds of measures may be of considerable interest and examples where they arise can be given.

Now let the kernel K be a non-negative, non-increasing real valued function on $[0, \propto)$, and with $\int_0^{\propto} K(z)dz = 1$. Let $p_{n,b}(x) = \sum_1^n bK(bd(x,X_i))/n$ where $b \geq 0$ is a parameter chosen by the user. For (x,y) in M^*, $R_{n,b}(x,y) = p_{n,b}(x)/p_{n,b}(y)$ is a plausible estimate of $R(x,y)$ and it is shown that as b and n increase without bound, $R_{n,b} \to R$ a.s.

It is intuitive that even when $\lim_{r \to 0} P(S(x,r))/P(S(y,r))$ is 0 or $+\propto$, $R_{n,b}$ will converge a.s. to 0 or $+\propto$ accordingly, but this situation will be treated elswhere.

It is also shown that R provides a workable theory of conditional probability in the general metric space context, without the technical complexities of the Radon-Nykodym approach. A few examples are given of the estimate $R_{n,b}$ illustrating the possible applications including application in psychology