CHAPTER 7

Small Sample Efficiency

7.1. Introduction. There is no small sample analogue of third order efficiency, but the same ideas can be used to derive exact results for small samples. Bickel, Götze and van Zwet (1985) use a Bayesian argument and perturbation of loss functions to prove the old result that $\bar{x}$ is a minimum variance unbiased estimate for a normal mean, without using the Cramér–Rao inequality or the Rao–Blackwell theorem. (The result is slightly weaker in that there is a restriction on the estimates considered.) We reproduce this argument in Section 7.4.

Consider the following scenario. You have a practical problem, the sample size n is reasonably large, say, $n = 50$, you know $\hat{\theta}$ is not only FOE but TOE, and simulations show asymptotic normality is well borne out by simulations. Out of this clear, blue sky appears an estimate T which is always at least 80% as efficient as $\hat{\theta}$, most of the time more efficient and occasionally more than *30 times more efficient than* $\hat{\theta}$. That this nightmare (to people who have come to adore the mle) can be a reality is documented in Khedr and Katti (1982).

Note that not only higher order optimality of $\hat{\theta}$, but even first order optimality of $\hat{\theta}$ seems to be in doubt. What has gone wrong?

It is a weakness of all asymptotic theories of optimality that they apply only to sequences of estimates and are silent about what can happen with a particular n (even if large) and a particular estimate. The situation is similar to that with asymptotic expansions which may be correct to $o(n^{-1})$, but may be pretty bad for a particular n, even if n is large. (This is not the case in the above example since simulation of the mean square of $\hat{\theta}$ seems to agree well with its asymptotic value.) This can happen for the trivial reason that a term $An^{-3/2}$ is $o(n^{-1})$, but may be large for n as large as 50 because the constant A is large. For most asymptotic expansions, we do not have good bounds on the constant A, but accuracy can be and often is checked by simulation. Also, one has the Berry–Esseen bound for asymptotic normality for $\sqrt{n}(\bar{x} - \theta)$