CHAPTER 5

Expansion of the Posterior, Bayes Estimate and Bayes Risk

5.1. Expansion of the posterior. Let $P(\theta)$ as well as $\pi(\theta)$ stand for a prior probability density and, deviating slightly from Ghosh, Sinha and Joshi [(1982), page 422]

$$b = -\frac{1}{n}\frac{d^2 \log p(X_1, X_2, \dots, X_n|\theta)}{d\theta^2}\bigg|_{\hat{\theta}}. \tag{5.1}$$

Let

$$F_n(h) = P\{\sqrt{nb}\,(\theta - \hat{\theta}) < h|X_1, X_2, \dots, X_n\} \tag{5.2}$$

be the posterior distribution function of the normalized θ. Under various conditions, $F_n(h)$ is approximately $\Phi(h)$, where Φ is the standard normal distribution function.

Here is a typical result. Assume regularity conditions on $p(x|\theta)$ and let $\pi(\theta)$ be continuous and positive at a fixed point θ_0. Then

$$\lim_{n\to\infty} \sup_h |F_n(h) - \Phi(h)| \to 0 \quad \text{a.s. } (P_{\theta_0}). \tag{5.3}$$

Le Cam (1958) has a similar theorem under P_π, where P_π is the marginal distribution of $\{X_n\}$ under $\pi \otimes P_\theta$. Here $\pi \otimes P_\theta$ stands for the joint distribution of θ and X's under which θ has density $\pi(\theta)$ and, given θ, X's have the joint distribution P_θ. Under P_θ, X's are i.i.d. $p(x|\theta)$.

Under stronger conditions on $p(x|\theta)$ and the assumption of $(k+1)$ times continuous differentiability of $\pi(\theta)$ at θ_0 and $P(\theta_0) > 0$, Johnson (1970) proves the following rigorous and precise version of a refinement due to Lindley (1961).