CHAPTER 4

Curvature and Information Loss

4.1. Curvature. This chapter contains a very informal discussion of curvature and connections along the lines of Efron (1975), who introduced curvature, and his discussant, Dawid (1975), who related Efron's curvature to the notion of "connection" in differential geometry. A more rigorous treatment is available in the articles in Amari, Barndorff-Nielsen, Kass, Lauritzen and Rao (1987); see also Kass (1989).

Consider a planar curve

$$y = y(x).$$

Then the curvature at x, $\gamma(x)$, is the rate at which the tangent changes direction in a neighborhood of x and is defined as

$$\gamma(x) = \frac{da}{ds} = \lim_{h \to 0} \frac{a(x+h) - a(x)}{s(x+h) - s(x)},$$

where $a(x)$ is the angle that the tangent makes with a fixed line and $s(x)$ is the length along the curve up to x from a fixed point.

Consider now a curved exponential, Θ an open interval and $\beta(\theta)$ a curve in R^k. Given two K-dimensional vectors, define the inner product by

$$\langle \beta_1, \beta_2 \rangle = \beta_1 \Sigma \beta_2^T,$$

where $\Sigma = \Sigma_{\theta_0}$ is a positive definite matrix to be specified later. Then,

$$\cos(\text{angle between } \beta_1, \beta_2) = \langle \beta_1, \beta_2 \rangle / \| \beta_1 \| \cdot \| \beta_2 \|, \tag{4.1}$$

where $\| \beta \| = (\beta \Sigma \beta^T)^{1/2}$.

If s_θ is the length of the curve $\beta = \beta(\theta)$ from θ_0 to θ, then

$$\left. \frac{ds(\theta)}{d\theta} \right|_{\theta_0} = \left(\dot{\beta}(\theta_0) \Sigma_{\theta_0} \dot{\beta}(\theta_0) \right)^{1/2},$$