CHAPTER 3

Third Order Efficiency for Curved Exponentials

3.1. The main result. Consider a curved exponential as introduced in the previous chapter, that is, except for a factor involving x only,

$$p(x|\theta) = c(\theta)\exp\left\{\sum_{i=1}^{k} \beta_i(\theta) f_i(x)\right\}.$$

For the time being, $\Theta \subset R$. Extensions to higher dimensions will be considered briefly later in the chapter. Recall also that $Z_{ij} = f_i(x_j)$, $Z_j = (Z_{1j}, \dots, Z_{kj})$, $\bar{Z} = n^{-1}\Sigma_1^n Z_j$, $\mu_i(\theta) = E(Z_{ij}|\theta)$, $\mu = (\mu_1, \dots, \mu_k)_\theta$ and $[\sigma_{ii'}(\theta)]$ is the dispension matrix of Z_j. Moreover, $\beta(\theta)$ lies in the interior of the natural parameter space of the multiparameter exponential, which is, apart from a factor of x,

$$p(x|\beta) = d(\beta)\exp\left\{\sum \beta_i f_i(x)\right\}.$$

So $p(x|\beta)$ is (real) analytic in β, and, hence, by our assumption of thrice continuous differentiability of $\beta(\theta)$, it follows that $p(x|\theta)$ is thrice continuously differentiable in θ. Also by assumption, $1, f_1, f_2, \dots, f_k$ are linearly independent, so that, among other things, $[\sigma_{ii'}]$ is positive definite.

Fisher consistent estimates T_n are of the form

$$T_n = H(\bar{Z}),$$

where $H(\mu(\theta)) = \theta$ and H is thrice continuously differentiable in a neighborhood of $\mu(\theta)$ for all θ. In Section 2.5 the mle was exhibited as a Fisher consistent estimate.

We will use frequently the calculations of Ghosh and Subramanyam (1974). So it is convenient to occasionally use the notations there, namely, p or p^n for $\bar{Z}$ for π for μ. Of the three interpretations for expansions considered in Section 2.7, the third is the most convenient, and so we also introduce a