CHAPTER 2

Structural Features

This chapter is devoted to developing a mathematical understanding of the structures that are inherent to the mixture model, ranging from the simple properties of moments up to rather complicated features of exponential family mixtures. Sections 2.1 and 2.2 contain the material of greatest practical importance because they address features of the mixture model useful for diagnostic purposes. The material thereafter is very important for understanding the issues of identifiability of the latent distribution Q, but can be skimmed and returned to as needed for the later chapters.

2.1. Descriptive features.

2.1.1. *Some simple moment results.* One of the nicest mathematical features of the mixture model is the simple way in which the latent distribution Q enters into the calculation of expectation. Simply by reordering the order of integration (or summation), we obtain the fact that if $t(x)$ has expectation $\tau(\phi)$ under the unicomponent model $f(x;\phi)$, then it has expectation $\int \tau(\phi)\,dQ(\phi)$ under the mixture model $f(x;Q)$. This is easily shown using the latent variable Φ:

$$E[t(X);Q] = E[E[t(X)|\Phi]] = E[\tau(\Phi)].$$

Using the latent variable also simplifies the calculation of variances under the mixture model:

$$\text{Var}[t(X);Q] = \text{Var}(E[t(X)|\Phi]) + E(\text{Var}[t(X)|\Phi]). \tag{2.1}$$

To illustrate these formulas, suppose that X comes from a mixture of Poisson densities with mean parameter ϕ. Then the following simple relationships between the marginal mean and variance of X and the latent variable Φ hold:

$$\begin{aligned} E(X;Q) &= E[\Phi], \\ \text{Var}(X;Q) &= \text{Var}[\Phi] + E[\Phi]. \end{aligned}$$

[*Exercise.*] Manipulation of these equations then shows that the variance of X in a Poisson mixture model is inflated, compared to a unicomponent Poisson