LECTURE 8

Examples of Long-Range Dependence

An early example of a process that might be termed long-range dependent was given in Rosenblatt (1961). The process $\{X_k\}$ is a quadratic function

$$X_k = Y_k^2 - 1$$

of a Gaussian stationary sequence Y_k with $EY_k \equiv 0$ and covariance sequence

$$r_k = (1 + k^2)^{-\gamma}, \qquad \gamma > 0.$$

The spectral density $g(\lambda)$ of the process Y_k is continuous and bounded away from zero if $|\lambda| > \varepsilon > 0$. Further, if $\gamma < \frac{1}{2}$, g has a singularity of the form $|\lambda|^{2\gamma - 1}$ in the neighborhood of $\lambda = 0$. The covariance sequence of X_k is $r_k = 2(1 + k^2)^{-2\gamma}$. Let $\gamma < \frac{1}{4}$. Then the spectral density $f(\lambda)$ of $\{X_k\}$ has a singularity of the form $|\lambda|^{4\gamma - 1}$ in the vicinity of $\lambda = 0$. Actually this family of processes was constructed to give simple examples of stationary sequences that are not strongly mixing. One can show that

$$n^{-1+2\gamma} \sum_{k=1}^{n} X_k$$

has a non-Gaussian limiting distribution as $n \to \infty$. This implies that the sequence $\{X_k\}$ is not strongly mixing for if it were, by the theorem of Rosenblatt (1956b), the limiting distribution would have to be normal since all the assumptions other than strong mixing are satisfied. The process $\{X_k\}$ would today be called long-range dependent because the partial sums have a non-Gaussian (and nonstable) limiting distribution. The non-Gaussian limiting behavior is easy to exhibit. The characteristic function of the normalized partial sum is

$$|I - 2itn^{-1+2\gamma}R|^{-1/2} \exp\{-in^{2\gamma}t\} = \exp\left\{\frac{1}{2}\sum_{k=2}^{\infty} (2itn^{-1+2\gamma})^k \operatorname{sp}(R^k)/k\right\}$$