LECTURE 7

Spectral Densities and Cumulants

We have already remarked on the theorem of Herglotz and what it implies for the representation of the covariance function of a stationary process X_k with finite second moments. Assume, for convenience, that $EX_k \equiv 0$. There is a result due to Cramér which gives a parallel representation of the process X_k itself as a Fourier–Stieltjes stochastic integral of a random process with orthogonal increments $z(\lambda)$,

$$(7.1) \qquad Ez(\lambda) \equiv 0, \qquad E\,dz(\lambda)\,\overline{dz(\mu)} = \delta(\lambda - \mu)\,dG(\lambda)$$

with δ the Kronecker delta

$$\delta(\lambda) = \begin{cases} 1 & \text{if } \lambda = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If the process is real-valued,

$$dz(\lambda) = \overline{dz(-\lambda)},$$
$$dG(\lambda) = dG(-\lambda).$$

The representation of X_k is

$$X_k = \int_{-\pi}^{\pi} e^{ik\lambda}\,dz(\lambda).$$

Knowledge of the spectral distribution function or its derivative $g(\lambda)$ (assuming G absolutely continuous) is clearly of interest in a host of linear problems or Gaussian problems. However, in case of nonlinearity or of non-Gaussian character, higher order moments (assuming they exist) can convey additional information. Let

$$\varphi(t_1, \dots, t_k) = E\exp\left\{ i\sum_{j=1}^{k} t_j X_j \right\} = \varphi(t).$$