SECTION 12

Random Convex Sets

Donoho (1982) and Donoho and Gasko (1987) studied an operation proposed by Tukey for extending the idea of trimming to multidimensional data. Nolan (1989a) gave a rigorous treatment of the asymptotic theory. Essentially the arguments express the various statistics of interest as differentiable functionals of an empirical measure. The treatment in this section will show how to do this without the formal machinery of compact differentiability for functionals, by working directly with almost sure representations. [Same amount of work, different packaging.]

To keep the discussion simple, let us consider the case of an independent sample $\xi_1, \xi_2, \ldots$ of random vectors from the symmetric bivariate normal distribution P on $\mathbb{R}^2$, and consider only the analogue of 25% trimming.

The notation will be cleanest when expressed (using traditional empirical process terminology) in terms of the *empirical measure* P_n, which puts mass $1/n$ at each of the points $\xi_1(\omega), \ldots, \xi_n(\omega)$.

Let $\mathcal{H}$ denote the class of all closed halfspaces in $\mathbb{R}^2$. Define a random compact, convex set $K_n = K_n(\omega)$ by intersecting all those halfspaces that contain at least $3/4$ of the observations:

$$K_n(\omega) = \bigcap \{H \in \mathcal{H} : P_n H \geq \tfrac{3}{4}\}.$$

It is reasonable to hope that K_n should settle down to the set

$$B(r_0) = \bigcap \{H \in \mathcal{H} : PH \geq \tfrac{3}{4}\},$$

which is a closed ball centered at the origin with radius r_0 equal to the 75% point of the one-dimensional standard normal distribution. That is, if Φ denotes the $N(0,1)$ distribution function, then $r_0 = \Phi^{-1}(3/4) \approx .675$. Indeed, a simple continuity argument based on a uniform strong law of large numbers,

$$\sup_{\mathcal{H}} |P_n H - PH| \to 0 \qquad \text{almost surely,} \tag{12.1}$$

would show that, for each $\epsilon > 0$, there is probability one that

$$B(r_0 - \epsilon) \subseteq K_n(\omega) \subseteq B(r_0 + \epsilon) \qquad \text{eventually.}$$