SECTION 10

Functional Central Limit Theorems

When does the standardized partial-sum processes converge in distribution, in the sense of the previous section, to a Gaussian process with nice sample paths? This section will establish a workable sufficient condition.

Part of the condition will imply finiteness (almost everywhere) of the envelope functions, which will mean that $S_n(\omega, \cdot)$ is a bounded function on T, for almost all ω. Ignoring negligible sets of ω, we may therefore treat S_n as a random element of the space $B(T)$ of all bounded, real-valued functions on T. The natural metric for this space is given by the uniform distance,

$$d(x, y) = \sup_t |x(t) - y(t)|.$$

One should take care not to confuse d with any metric, or pseudometric, ρ defined on T. Usually such a ρ will have something to do with the covariance structure of the partial-sum processes. The interesting limit distributions will be Gaussian processes that concentrate on the set

$$U_\rho(T) = \{x \in B(T) : x \text{ is uniformly } \rho \text{ continuous}\}.$$

Under the uniform metric d, the space $U_\rho(T)$ is separable if and only if T is totally bounded under ρ. [Notice that total boundedness excludes examples such as the real line under its usual metric.] In the separable case, a Borel probability measure P on $U_\rho(T)$ is uniquely determined by its finite dimensional projections,

$$P(B \mid t_1, \ldots, t_k) = P\{x \in U_\rho(T) : (x(t_1), \ldots, x(t_k)) \in B\},$$

with $\{t_1, \ldots, t_k\}$ ranging over all finite subsets of T and B ranging over all Borel sets in $\mathbb{R}^k$, for $k = 1, 2, \ldots$.

Let us first consider a general sequence of stochastic processes indexed by T,

$$\{X_n(\omega, t) : t \in T\} \qquad \text{for } n = 1, 2, \ldots,$$