SECTION 9

Convergence in Distribution and Almost Sure Representation

Classical limit theorems for sums of independent random vectors will often suggest a standardization for the partial-sum process, S_n, so that its finite dimensional projections have a limiting distribution. It is then natural to ask whether the standardized stochastic process also has a limiting distribution, in some appropriate sense. The traditional sense has been that of a functional limit theorem. One identifies some metric space of real-valued functions on T that contains all the standardized sample paths, and then one invokes a general theory for convergence in distribution of random elements of a metric space (or weak convergence of probability measures on the space).

For example, if $T = [0,1]$ and the sample paths of S_n have only simple discontinuities, the theory of weak convergence for $D[0,1]$ might apply.

Unfortunately, even for such simple processes as the empirical distribution function for samples from the Uniform$[0,1]$ distribution, awkward measurability complications arise. With $D[0,1]$ either one skirts the issue by adopting a Skorohod metric, or one retains the uniform metric at the cost of some measure theoretic modification of the definition of convergence in distribution.

For index sets more complicated than $[0,1]$ there is usually no adequate generalization of the Skorohod metric. The measurability complications cannot be defined away. One must face the possibility that the expectations appearing in plausible definitions for convergence in distribution need not be well defined. Of the numerous general theories proposed to handle this problem, the one introduced by Hoffmann-Jørgensen (and developed further by Dudley 1985) is undoubtedly the best. It substitutes outer expectations for expectations. It succeeds where other